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A new generalization of Aczél's inequality and its applications to an improvement of Bellman's inequality

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Abstract

A new generalized version of Aczél's inequality is proved. This is a unified generalization of some known results. Moreover, the result is applied to the improvement of the well-known Bellman's inequality.

Keywords: Aczél's inequality; Bellman's inequality; Power means inequality; Čebyšev's inequality; Generalization

1. Introduction

In 1956, Aczél [1] proved the following result:

$$\left(a_1^2 - \sum_{i=2}^{n} a_i^2\right)\left(b_1^2 - \sum_{i=2}^{n} b_i^2\right) \le \left(a_1 b_1 - \sum_{i=2}^{n} a_i b_i\right)^2, \tag{1}$$

where a_i, b_i $(i = 1, 2, \ldots, n)$ are positive numbers such that $a_1^2 - \sum_{i=2}^{n} a_i^2 > 0$ or $b_1^2 - \sum_{i=2}^{n} b_i^2 > 0$. This inequality is called Aczél's inequality.

It is well known that Aczél's inequality has important applications in the theory of functional equations in non-Euclidean geometry. In recent years, considerable attention has been given to this inequality involving its generalizations, variations and applications (see [2–11] and references therein). We state here some improvements of Aczél's inequality.

Popoviciu [12] first presented an exponential extension which is stated in the following theorem.

Theorem A. *Let $p > 0$, $q > 0$, $p^{-1} + q^{-1} = 1$, and let a_i, b_i $(i = 1, 2, \ldots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ and $b_1^q - \sum_{i=2}^{n} b_i^q > 0$. Then*

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$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \le a_1 b_1 - \sum_{i=2}^n a_i b_i. \tag{2}$$

Vasić and Pečarić [13] established a further extension of inequality (2) as follows:

Theorem B. *Let* $p_j > 0\ (j = 1, 2, \ldots, m)$, $p_1^{-1} + p_2^{-1} + \cdots + p_m^{-1} \ge 1$, *and let* $a_{ij}\ (i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ *be positive numbers such that* $a_{1j}^{p_j} - \sum_{i=2}^n a_{ij}^{p_j} > 0$ *for* $j = 1, 2, \ldots, m$. *Then*

$$\prod_{j=1}^m \left(a_{1j}^{p_j} - \sum_{i=2}^n a_{ij}^{p_j}\right)^{\frac{1}{p_j}} \le \prod_{j=1}^m a_{1j} - \sum_{i=2}^n \prod_{j=1}^m a_{ij}. \tag{3}$$

In a recent paper [14], Wu and Debnath generalized inequality (3) in the following form:

Theorem C. *Let* p_j *and* $a_{ij}\ (i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ *be positive numbers such that* $a_{1j}^{p_j} - \sum_{i=2}^n a_{ij}^{p_j} > 0$ *for* $j = 1, 2, \ldots, m$. *Then*

$$\prod_{j=1}^m \left(a_{1j}^{p_j} - \sum_{i=2}^n a_{ij}^{p_j}\right)^{\frac{1}{p_j}} \le n^{1-\min\{p_1^{-1}+p_2^{-1}+\cdots+p_m^{-1},1\}} \prod_{j=1}^m a_{1j} - \sum_{i=2}^n \prod_{j=1}^m a_{ij}. \tag{4}$$

In this work, we give a new generalized version of Aczél's inequality which is a unified generalization of several results of previous papers [12–14]. Finally, we provide an application to the improvement of the well-known Bellman's inequality.

2. Generalization of Aczél's inequality

Theorem 1. *Let* p_j *and* $a_{ij}\ (i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ *be positive numbers, and let* $k\ (1 \le k < n)$ *be a positive integer such that* $a_{1j} \le a_{2j} \le \cdots \le a_{kj}$ *(or* $a_{1j} \ge a_{2j} \ge \cdots \ge a_{kj}$*) and* $\sum_{i=1}^k a_{ij}^{p_j} - \sum_{i=k+1}^n a_{ij}^{p_j} > 0$ *for* $j = 1, 2, \ldots, m$. *Then we have the inequality*

$$\prod_{j=1}^m \left(\sum_{i=1}^k a_{ij}^{p_j} - \sum_{i=k+1}^n a_{ij}^{p_j}\right)^{\frac{1}{p_j}} \le (n-k+1)^{1-\min\{p_1^{-1}+p_2^{-1}+\cdots+p_m^{-1},1\}} k^{p_1^{-1}+p_2^{-1}+\cdots+p_m^{-1}-\min\{(p_1+p_2+\cdots+p_m)^{-1},1\}} \times \sum_{i=1}^k \prod_{j=1}^m a_{ij} - \sum_{i=k+1}^n \prod_{j=1}^m a_{ij}. \tag{5}$$

In order to prove Theorem 1, we need the following lemmas.

Lemma 1. *Let* $a_i > 0\ (i = 1, 2, \ldots, n)$ *and* $p > 0$. *Then*

$$\sum_{i=1}^n a_i^p \le n^{1-\min\{p,1\}} \left(\sum_{i=1}^n a_i\right)^p. \tag{6}$$

Lemma 2. *Let* $a_{ij}\ (i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ *be real numbers such that* $a_{1j} \le a_{2j} \le \cdots \le a_{nj}$ *(or* $a_{1j} \ge a_{2j} \ge \cdots \ge a_{nj}$*) for* $j = 1, 2, \ldots, m$. *Then*

$$\prod_{j=1}^m \sum_{i=1}^n a_{ij} \le n^{m-1} \sum_{i=1}^n \prod_{j=1}^m a_{ij}. \tag{7}$$

Inequalities (6) and (7) are known as the power means inequality and Čebyšev's inequality respectively (see Mitrinović [8]).

We now prove Theorem 1.

Proof of Theorem 1. In view of the assumption that $p_j > 0, a_{ij} > 0$ and

$$\sum_{i=1}^{k} a_{ij}^{p_j} - \sum_{i=k+1}^{n} a_{ij}^{p_j} > 0 \quad (i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m),$$

it follows from Theorem C that

$$\prod_{j=1}^{m} \left(\sum_{i=1}^{k} a_{ij}^{p_j} - \sum_{i=k+1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} \le (n-k+1)^{1-\min\{p_1^{-1}+p_2^{-1}+\cdots+p_m^{-1},1\}} \prod_{j=1}^{m} \left(\sum_{i=1}^{k} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} - \sum_{i=k+1}^{n} \prod_{j=1}^{m} a_{ij}. \tag{8}$$

On the other hand, using the power means inequality (6) and Čebyšev's inequality (7) respectively, we have

$$\begin{aligned}
\prod_{j=1}^{m} \left(\sum_{i=1}^{k} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} &= \prod_{j=1}^{m} \left[\sum_{i=1}^{k} \left(a_{ij}^{p_1+p_2+\cdots+p_m} \right)^{\frac{p_j}{p_1+p_2+\cdots+p_m}} \right]^{\frac{1}{p_j}} \\
&\le \prod_{j=1}^{m} \left[k^{1-\frac{p_j}{p_1+p_2+\cdots+p_m}} \left(\sum_{i=1}^{k} a_{ij}^{p_1+p_2+\cdots+p_m} \right)^{\frac{p_j}{p_1+p_2+\cdots+p_m}} \right]^{\frac{1}{p_j}} \\
&= \prod_{j=1}^{m} \left[k^{\frac{1}{p_j}-\frac{1}{p_1+p_2+\cdots+p_m}} \left(\sum_{i=1}^{k} a_{ij}^{p_1+p_2+\cdots+p_m} \right)^{\frac{1}{p_1+p_2+\cdots+p_m}} \right] \\
&= k^{\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}-\frac{m}{p_1+p_2+\cdots+p_m}} \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{k} a_{ij}^{p_1+p_2+\cdots+p_m} \right) \right]^{\frac{1}{p_1+p_2+\cdots+p_m}} \\
&\le k^{\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}-\frac{m}{p_1+p_2+\cdots+p_m}} \left[k^{m-1} \sum_{i=1}^{k} \left(\prod_{j=1}^{m} a_{ij}^{p_1+p_2+\cdots+p_m} \right) \right]^{\frac{1}{p_1+p_2+\cdots+p_m}} \\
&= k^{\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}-\frac{1}{p_1+p_2+\cdots+p_m}} \left[\sum_{i=1}^{k} \left(\prod_{j=1}^{m} a_{ij}^{p_1+p_2+\cdots+p_m} \right) \right]^{\frac{1}{p_1+p_2+\cdots+p_m}} \\
&\le k^{\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}-\frac{1}{p_1+p_2+\cdots+p_m}} \\
&\quad \times \left[k^{1-\min\{p_1+p_2+\cdots+p_m,1\}} \left(\sum_{i=1}^{k} \prod_{j=1}^{m} a_{ij} \right)^{p_1+p_2+\cdots+p_m} \right]^{\frac{1}{p_1+p_2+\cdots+p_m}} \\
&= k^{p_1^{-1}+p_2^{-1}+\cdots+p_m^{-1}-\min\{(p_1+p_2+\cdots+p_m)^{-1},1\}} \sum_{i=1}^{k} \prod_{j=1}^{m} a_{ij}.
\end{aligned} \tag{9}$$

Combining (8) and (9) leads to the desired inequality (5). This completes the proof of Theorem 1. □

Remark 1. Putting $k = 1$ in (5) yields immediately the inequality (4) asserted by Theorem C. Further, putting $k = 1$ and $p_1^{-1} + p_2^{-1} + \cdots + p_m^{-1} \ge 1$ in (5) leads to the result of Theorem B.

If we put in Theorem 1 $m = 2$, $p_1 = p$, $p_2 = q$, $a_{i1} = a_i$, $a_{i2} = b_i$ $(i = 1, 2, \ldots, n)$, we obtain the following corollary:

Corollary 1. *Let p, q, a_i, b_i $(i = 1, 2, \ldots, n)$ be positive numbers, and let k $(1 \le k < n)$ be a positive integer such that $\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p > 0$, $\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q > 0$ and the sequences $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are monotonic in the same direction. Then we have the inequality*

$$\left(\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}}$$
$$\le (n-k+1)^{1-\min\{p^{-1}+q^{-1},1\}} k^{p^{-1}+q^{-1}-\min\{(p+q)^{-1},1\}} \sum_{i=1}^{k} a_i b_i - \sum_{i=k+1}^{n} a_i b_i. \tag{10}$$

In particular, letting $p^{-1} + q^{-1} = 1$ in (10) gives

Corollary 2. *Let p, q, a_i, b_i $(i = 1, 2, \ldots, n)$ be positive numbers with $p^{-1} + q^{-1} = 1$, and let k $(1 \le k < n)$ be a positive integer such that $\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p > 0$, $\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q > 0$ and the sequences $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are monotonic in the same direction. Then we have the inequality*

$$\left(\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} \le k^{(pq-1)/pq} \sum_{i=1}^{k} a_i b_i - \sum_{i=k+1}^{n} a_i b_i. \tag{11}$$

Corollary 2 with a special case $k = 1$ yields Popoviciu's inequality (2).

3. Application to improvement of Bellman's inequality

The following celebrated inequality is known as Bellman's inequality (see Bellman [15]):

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{\frac{1}{p}} + \left(b_2^p - \sum_{i=2}^{n} b_i^p\right)^{\frac{1}{p}} \le \left((a_1+b_1)^p - \sum_{i=2}^{n} (a_i+b_i)^p\right)^{\frac{1}{p}}, \tag{12}$$

where $p \ge 1$, a_i, b_i $(i = 1, 2, \ldots, n)$ are positive numbers such that $a_i^p - \sum_{i=2}^{n} a_i^p > 0$ and $b_i^p - \sum_{i=2}^{n} b_i^p > 0$.

We give here an improvement of Bellman's inequality, that is,

Theorem 2. *Let $p \ge 1$, let k $(1 \le k < n)$ be a positive integer, and let a_i, b_i $(i = 1, 2, \ldots, n)$ be positive numbers such that $\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p > 0$, $\sum_{i=1}^{k} b_i^p - \sum_{i=k+1}^{n} b_i^p > 0$ and the sequences $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are monotonic in the same direction. Then we have the inequality*

$$\left(\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{k} b_i^p - \sum_{i=k+1}^{n} b_i^p\right)^{\frac{1}{p}} \le \left(k^{(p^2-p+1)/p} \sum_{i=1}^{k} (a_i+b_i)^p - \sum_{i=k+1}^{n} (a_i+b_i)^p\right)^{\frac{1}{p}}. \tag{13}$$

Proof. When $p = 1$, (13) is an identity. We suppose $p > 1$ below.

Note that the sequences $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are monotonic in the same direction, without loss of generality we assume that $a_1 \le a_2 \le \cdots \le a_k, b_1 \le b_2 \le \cdots \le b_k$.

From the assumption

$$\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p > 0, \qquad \sum_{i=1}^{k} b_i^p - \sum_{i=k+1}^{n} b_i^p > 0,$$

we find

$$a_k \ge \left(\frac{1}{k}\sum_{i=1}^{k} a_i^p\right)^{\frac{1}{p}} > \left(\frac{1}{k}\sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}}, \qquad b_k \ge \left(\frac{1}{k}\sum_{i=1}^{k} b_i^p\right)^{\frac{1}{p}} > \left(\frac{1}{k}\sum_{i=k+1}^{n} b_i^p\right)^{\frac{1}{p}}. \tag{14}$$

Applying inequality (14) and Minkowski's inequality (see Mitrinović [8]), we obtain

$$k^{(p^2-p+1)/p}\sum_{i=1}^{k}(a_i+b_i)^p \geq k^{(p^2-p+1)/p}(a_k+b_k)^p \geq k^{(p-1)^2/p}\left[\left(\sum_{i=k+1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=k+1}^{n}b_i^p\right)^{\frac{1}{p}}\right]^p$$

$$\geq k^{(p-1)^2/p}\sum_{i=k+1}^{n}(a_i+b_i)^p \geq \sum_{i=k+1}^{n}(a_i+b_i)^p. \tag{15}$$

By appealing to inequality (15) with the assumptions

$$a_1 \leq a_2 \leq \cdots \leq a_k, \qquad b_1 \leq b_2 \leq \cdots \leq b_k, \qquad \sum_{i=1}^{k}a_i^p - \sum_{i=k+1}^{n}a_i^p > 0, \qquad \sum_{i=1}^{k}b_i^p - \sum_{i=k+1}^{n}b_i^p > 0,$$

we now deduce from Corollary 2 that

$$\left(\sum_{i=1}^{k}a_i^p - \sum_{i=k+1}^{n}a_i^p\right)^{\frac{1}{p}}\left[k^{(p^2-p+1)/p}\sum_{i=1}^{k}(a_i+b_i)^p - \sum_{i=k+1}^{n}(a_i+b_i)^p\right]^{1-\frac{1}{p}}$$

$$\leq k^{(p^2-p+1)/p^2}\sum_{i=1}^{k}k^{(p-1)(p^2-p+1)/p^2}a_i(a_i+b_i)^{p-1} - \sum_{i=k+1}^{n}a_i(a_i+b_i)^{p-1}$$

$$= k^{(p^2-p+1)/p}\sum_{i=1}^{k}a_i(a_i+b_i)^{p-1} - \sum_{i=k+1}^{n}a_i(a_i+b_i)^{p-1}, \tag{16}$$

and

$$\left(\sum_{i=1}^{k}b_i^p - \sum_{i=k+1}^{n}b_i^p\right)^{\frac{1}{p}}\left[k^{(p^2-p+1)/p}\sum_{i=1}^{k}(a_i+b_i)^p - \sum_{i=k+1}^{n}(a_i+b_i)^p\right]^{1-\frac{1}{p}}$$

$$\leq k^{(p^2-p+1)/p}\sum_{i=1}^{k}b_i(a_i+b_i)^{p-1} - \sum_{i=k+1}^{n}b_i(a_i+b_i)^{p-1}. \tag{17}$$

Adding (16) and (17) gives

$$\left[\left(\sum_{i=1}^{k}a_i^p - \sum_{i=k+1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{k}b_i^p - \sum_{i=k+1}^{n}b_i^p\right)^{\frac{1}{p}}\right]\left[k^{(p^2-p+1)/p}\sum_{i=1}^{k}(a_i+b_i)^p - \sum_{i=k+1}^{n}(a_i+b_i)^p\right]^{1-\frac{1}{p}}$$

$$\leq k^{(p^2-p+1)/p}\sum_{i=1}^{k}(a_i+b_i)^p - \sum_{i=k+1}^{n}(a_i+b_i)^p, \tag{18}$$

which leads to the desired inequality (13). The proof of Theorem 2 is complete. □

Remark 2. Note that when $k = 1$, then inequality (13) becomes Bellman's inequality. So Bellman's inequality is just a special case of the inequality in Theorem 2.

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